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A Geometric Presentation of the 26-Dimensional Module for $F_4(q)$

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The fixed-point sheaf of the group $F_4(q)$, acting on its 26-dimensional module V, is described; it is shown that the 0-homology module is isomorphic to V. This gives a presentation of V by geometric generators and relations.

INTRODUCTION

We will prove the following result:

THEOREM. *Let V be the 26-dimensional "natural" module for $F_4(q)$,* and *$\mathscr{F}_V$ the irreducible presheaf it determines. Then $H_0(\mathscr{F}_V) \simeq V$.*

Before setting the background for the theorem we make three observations:

(1) For q of any characteristic except 3, V is irreducible [CPS, Table (4.5)] and $\mathscr{F}_V$ is the full fixed-point sheaf. In characteristic 3, V is indecomposable and has a trivial submodule with a 25-dimensional quotient W; then $\mathscr{F}_V$ is isomorphic to the full fixed-point sheaf of W.

(2) We note that V comes from the 27-dimensional exceptional Jordan algebra. The subspaces of V fixed by unipotent subgroups of $F_4(q)$ correspond to certain subspaces where the multiplication restricts to 0. We will not however make use of the algebra in the proof of the theorem.

(3) The finite field $\mathbf{F}_q$ can in fact be replaced by any perfect field of prime characteristic, or a field of algebraic numbers (as in [SV], which the proof requires).

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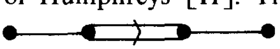
For group structure theory, and local group theory in particular, it is useful to identify group modules from local-geometric information. Much work has been done in this area by Ronan–Smith, Smith–Völklein, Segev–Smith, and Völklein [RS, SV, SS, V1]. In particular, given a (universal) Chevalley group acting on an irreducible module in the natural characteristic, we can recover the module as a quotient of the 0-homology module of the fixed-point presheaf for G [RS, Theorem (2.3)].

In the present situation we are considering $G \simeq F_4(q)$ acting on its 26-dimensional module V. We can exploit the Lie theory of G and the weight decomposition of V to reduce much of the proof of the theorem to smaller-rank cases, following the method of Völklein [V1] (where the main theorem reduces to consideration of subsystems of type A_2). The more difficult of these smaller cases have already been computed in some of the work mentioned above. The root system, the Weyl group, and the conjugacy classes of parabolic subgroups P of G will be described, as well as certain P-submodules and P-flags of V. With this information we give the structure of the building Δ and the fixed-point sheaf $\mathscr{F}_V$ for G on Δ. Our main computation involves exhibiting a generating set for $H_0(\mathscr{F}_V)$. To show that the set spans $H_0(\mathscr{F}_V)$, we employ the apartment method of Section 4 in the Ronan–Smith paper [RS]. This spanning argument will establish that $\dim(H_0(\mathscr{F}_V)) \leqslant 36$. We then show, using induction, that in fact 26 dimensions of generators are sufficient. As a corollary to the theorem, we obtain an independent proof of the calculation of the 1-cohomology of G acting on V done by Cline–Parshall–Scott [CPS].

PART 1: BACKGROUND INFORMATION AND THE FIXED POINT SHEAF

THE ROOT SYSTEM FOR F_4

For a reference for the standard facts here, see, for example, Part III of Humphreys [H]. The associated Dynkin diagram for F_4 is given by  corresponding to the Cartan matrix

$$\begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -2 & 0 \\ 0 & -1 & 2 & -2 \\ 0 & 0 & -1 & 2 \end{pmatrix}.$$

The irreducible root system Φ is of rank four. We denote a set of fundamental roots by $\Pi = \{1000, 0100, 0010, 0001\}$, where the position of the "1" corresponds to the node in the Dynkin diagram. Any root in Φ is expressed in the form $abcd = a(1000) + b(0100) + c(0010) + d(0001)$ with a, b, c, $d \in \mathbf{N}$ or a, b, c, $d \in -\mathbf{N}$. The complete set of positive roots (well known; e.g., Table 1 of [AS]) is listed here:

Short roots		Long roots		
0010	1110	1000	0100	2342
0110	1111	1100	0120	
0111	1121	1120	0122	
0121	1221	1220		
0011	1231	1122		
0001	1232	1222		
		1242		
		1342		

These along with their negatives compose the set Φ.

We are considering the action of $G = F_4(q)$ on the 26-dimensional irreducible module V which corresponds to high weight $\lambda := 1232$ (the highest short root). Under the G-action there is a weight-space decomposition of V into 24 one-spaces given by the 24 short roots and a 2-dimensional zero-weight space. This decomposition depends upon a choice of a Cartan subalgebra for the Lie algebra of G. For much of what follows, we will fix such a subalgebra and corresponding Cartan subgroup H.

Parabolic Subgroups and the Building of F_4

For a reference for this information see, for example, Chapters 2 and 15 of Carter [C]. The building Δ of type F_4 is a simplicial complex mirroring the structure of the lattice of parabolic subgroups of G. That is, if σ_1 and σ_2 are simplices of Δ and P_1 and P_2 are the corresponding parabolics in G, then σ_1 is a face of σ_2 if and only if $P_2 < P_1$. Then vertices in Δ correspond to maximal parabolic subgroups, while the highest-dimensional simplices correspond to Borel subgroups.

There are four conjugacy classes of maximal parabolic subgroups of G. We obtain representatives from the four subsets of Π of size three. Given any subset $\Pi' \subset \Pi$ we form the parabolic subgroup $P(\Pi') = \langle H, X_\alpha : \alpha \in \Pi$ or $-\alpha \in \Pi' \rangle$; here X_α denotes the root group of the root α. The group $P(\Pi')$ has a Levi decomposition as semidirect product of the unipotent radical $U(\Pi') = \langle X_\alpha : \alpha \in \{\Phi^+ \backslash \mathbf{N}\Pi'\} \rangle$ and a Levi complement $L(\Pi') = \langle H, X_\alpha : \alpha \in \{\mathbf{Z}\Pi' \cap \Phi\} \rangle$. (This decomposition depends of course on the choice of fundamental roots.)

We label the three-element subsets of Π as follows: $\Pi_1 = \Pi\backslash\{0001\}$; $\Pi_2 = \Pi\backslash\{0010\}$; $\Pi_3 = \Pi\backslash\{0100\}$; $\Pi_6 = \Pi\backslash\{1000\}$. The subscript $i = 1, 2, 3$, or 6 corresponds to the dimension of the smallest proper subspace of V that is stabilized by the parabolic $P(\Pi_i)$ or equivalently (compare [S]) the full fixed space of the unipotent radical $U(\Pi_i)$. We can directly check these dimensions using the fact that if α and β are roots (β short) with $\alpha \neq \beta$ and $\alpha + \beta$ not a short root, then X_α fixes the one-space V_β(the weight space of β). We find in fact that:

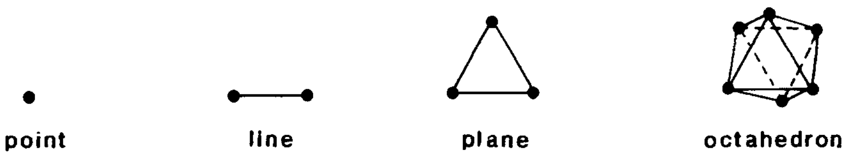
$U(\Pi_1)$ fixes V_{1232}

$U(\Pi_2)$ fixes V_{1232} and V_{1231}

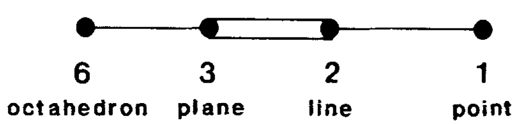
$U(\Pi_3)$ fixes V_{1232}, V_{1231}, and V_{1221}

$U(\Pi_6)$ fixes V_{1232}, V_{1231}, V_{1221}, V_{1121}, V_{1111}, and V_{1110}.

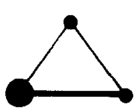
These spaces (and their G-conjugates) correspond to the four types of vertices in the building. We refer to them as points, lines, planes, and octahedra (respectively), for it is natural to represent each such space by a geometrical figure on the root-subspaces just listed:



Each root that was removed from Π to form Π_i ($i = 1, 2, 3, 6$) corresponds to a node in the Dynkin diagram. In this way there is a natural correspondence between the Dynkin diagram and the four classes of maximal parabolics. We express this in the diagram by:



The higher-order simplices arise by intersecting maximal parabolics which share a common Borel subgroup. The smaller parabolics stabilize flags of subspaces. For example, $P(\Pi_1) \cap P(\Pi_2)$ stabilizes the one space V_{1232} in the two-space $V_{1232} + V_{1231}$. Geometrically this is a point on a line:  Intersecting again gives a higher dimensional simplex, e.g., $P(\Pi_1) \cap P(\Pi_2) \cap P(\Pi_3)$ stabilizes a one-space in a two-space in a three-space; a point on a line on a plane:



Using these intersections we determine 15 proper parabolics; through conjugation by the subgroup N (cf. Chapter 7 of [C]) we obtain all of the simplices in an apartment, which we denote by A. The apartment has the structure of the Coxeter complex for the Weyl group W of type F_4 (see Chapter 15 of [C]).

THE FIXED-POINT SHEAF

We form the fixed-point sheaf $\mathscr{F}_V$ by attaching to each simplex the subspace of V fixed by the unipotent radical of the parabolic defining that simplex. For a vertex, this will correspond to the space of dimension 1, 2, 3, or 6 stabilized by the defining maximal parabolic as described above. For higher order simplices, the parabolic will stabilize a flag of subspaces; in this case we attach the smallest subspace in the flag. We denote the space attached to the simplex σ by V_σ.

There are natural linear maps connecting V_{σ_1} to V_{σ_2} whenever σ_2 is a face of σ_1. For if P_1 and P_2 are the corresponding parabolics then $P_1 \subset P_2$ and therefore P_1 stabilizes a flag F_1, which contains the flag F_2, stabilized by P_2; then necessarily the smallest subspace of F_1 is contained in the smallest one of F_2. The natural inclusion map induces a linear map from V_{σ_1} to V_{σ_2}.

We form a chain complex by composing the usual boundary chain map with these connecting inclusion maps. The homology quotients are formed in the usual way [RS, Sect. 1]. Our concern is with the bottom homology module $H_0(\mathscr{F}_V)$; the main effort in the paper is to establish its dimension.

PART 2: PROOF OF THE THEOREM

INTRODUCTION

Our main result is that the bottom homology module, which we will denote $\hat{V}$, is isomorphic to V. Here is an overview of the proof: Since V arises as a quotient of $\hat{V}$, it is sufficient to show that $\dim(\hat{V}) \leqslant 26$. For an apartment A, we construct a sum of one-spaces $L(A)$ which includes all one-spaces for short-root vertices on A. Since $\hat{V}$ is generated by vertex one-spaces (cf. [RS, (3.5)]), then $\hat{V}$ is spanned by the sum of these $L(A)$. We show in fact that $\hat{V} = L(A)$ for a fixed apartment by showing that $L(A') \subset L(A)$ for any apartment A' in Δ. Since Δ is connected under the "root" relation (cf. [RS, (4.4)]), we may assume that A' is a neighboring apartment to A. We then reduce the spanning argument for the neighbors to rank-2 considerations via Völklein [V1]. Most of the work in the proof is in establishing which rank-2 subsystems can occur in this context. Once they are identified, we can show that their generators can be found in $L(A)$ by appealing to results in the literature. This analysis will also establish that $\dim(L(A)) \leqslant 36$. To complete the proof we refine the construction of $L(A)$ to show it is at most 26-dimensional again relying on rank-2 analysis.

DEFINITIONS

Given an apartment A fixed by Cartan subgroup H we define the following ($\Phi(A)$ denotes the roots of A, X_α is the root group of the root α):

subgroups of G

$L_\alpha := \langle X_\alpha, X_{-\alpha} \rangle$

$L_{\alpha,\beta} := \langle L_\alpha, L_\beta \rangle$

subspaces of $\hat{V}$

$L(V_\alpha) := \langle V_\alpha^g : g \in L_\alpha \rangle$

$L_{\alpha,\beta}(V_\alpha) := \langle V_\alpha^g : g \in L_{\alpha,\beta} \rangle$

$L(A) := \langle L(V_\alpha)$: *$\alpha$ a short root in $\Phi(A)$*$\rangle$.

In these latter three constructions we have identified the sheaf 1-spaces V_α with their images in $\hat{V}$.

REDUCTION TO RANK 2

As noted in the Introduction to this section, in order to show $\hat{V} = L(A)$ it is sufficient to show that $L(A') \subset L(A)$ for A' a neighbor to A. Let A' be such a neighbor; then $\exists \beta \in \Phi(A)$, such that $A' = A^{g_\beta}$ for some $g_\beta \in L_\beta$, and therefore $L(A') = L(A)^{g_\beta} = \langle L(V_\alpha)^{g_\beta}$: α short in $\Phi(A)\rangle$. We must show that $L(V_\alpha)^{g_\beta} \subset L(A)$ for each $\alpha \in \Phi(A)$. Since $g_\beta \in L_\beta$, then $L(V_\alpha)^{g_\beta} \subset L_{\alpha,\beta}(V_\alpha)$. Then the problem becomes determining what possible subsystems of the type $L_{\alpha,\beta}(V_\alpha)$ with α short can occur.

ORBITS OF W_{1232}

To simplify the identification of the rank-2 subsystems we first determine the conjugacy classes under the Weyl group W of pairs of roots (α, β) where α is short. If we assume that $\alpha := \lambda := 1232$, the high short root, then β will lie in some orbit of the stabilizer subgroup, $W_\lambda = \{w \in W: \lambda^w = \lambda\}$. The conjugacy classes of pairs will correspond to these orbits, which we now determine by direct computation.

There are four fundamental reflections $\sigma_1, \sigma_2, \sigma_3$, and σ_4 which generate W. These act on the fundamental roots as

$$
\begin{array}{llll}
\sigma_1: 1000 \to -1000 & \sigma_2: 1000 \to 1100 & \sigma_3: 1000 & \sigma_4: 1000 \\
 0100 \to 1100 & 0100 \to -0100 & 0100 \to 0120 & 0100 \\
 0010 & 0010 \to 0110 & 0010 \to -0010 & 0010 \to 0011 \\
 0001 & 0001 & 0001 \to 0011 & 0001 \to -0001.
\end{array}
$$

For $i = 1, 2, 3$ we find $(\lambda, \alpha_i) = 0$; it follows that σ_i fixes λ. Then the (maximal) subgroup of W generated by σ_1, σ_2, and σ_3 is contained in W_λ; indeed $W_\lambda = \langle \sigma_1, \sigma_2, \sigma_3 \rangle$ since not all of W fixes λ.

LEMMA 1. *The short roots fall into five orbits under the action of* W_λ, *corresponding to inner product* $-2, -1, 0, 1$, *and* 2 *with* λ.

Proof. Since W preserves inner products there must be at least these five classes; we simply check that they split up no further:

(i) By definition of W_λ we have that λ is fixed as well as $-\lambda$ since the Weyl-group action commutes with the field. We thus obtain two single-element orbits corresponding to inner products 2 and -2.

(ii) $(1232, 0001) = 1 \cdot 0 + 2 \cdot 0 + 3(-1) + 2(2) = 1$. By checking the action of the fundamental reflections we obtain

$$
\begin{tikzcd}
0001 \arrow[r, "\sigma_3"] & 0011 \arrow[r, "\sigma_2"] & 0111 \arrow[r, "\sigma_1"] & 1111 \arrow[r, "\sigma_3"] & 1121 \arrow[r, "\sigma_1"] \arrow[d, "\sigma_2"] & 0121 \\
& & & & 1221 \arrow[r, "\sigma_3"] & 1231.
\end{tikzcd}
$$

As the action of W commutes with -1 we get the negatives of these roots in an orbit and these all have inner product -1 with 1232.

(iii) $(1232, 0010) = 1 \cdot 0 + 2(-2) + 3 \cdot 2 + 2(-1) = 0$. We get the orbit

$$
\begin{tikzcd}
0010 \arrow[r, "\sigma_2"] \arrow[d, "\sigma_3"] & 0110 \arrow[r, "\sigma_1"] & 1110 \\
-0010 \arrow[r, "\sigma_2"] & -0110 \arrow[r, "\sigma_1"] & 1110.
\end{tikzcd}
$$

We have now accounted for all 24 short roots in these five orbits so the lemma follows. ∎

LEMMA 2. *The long roots fall into three orbits under the action of W_λ, corresponding to inner products -2, 0, and 2 with λ.*

Proof. Observe $(1232,1000) = 1 \cdot 2 + 2 \cdot (-1) + 3 \cdot 0 + 2 \cdot 0 = 0$, and check

$$\begin{array}{cccc}
 & 0100 \xrightarrow{\sigma_3} & 0120 & \\
 & \uparrow \sigma_1 & & \\
1000 \xrightarrow{\sigma_2} & 1100 \xrightarrow{\sigma_3} & 1120 \xrightarrow{\sigma_2} & 1220 \\
\downarrow \sigma_1 & & & \\
-1000 \xrightarrow{\sigma_2} & -1100 \xrightarrow{\sigma_3} & -1120 \xrightarrow{\sigma_2} & -1220 \\
 & \downarrow \sigma_1 & & \\
 & -0100 \xrightarrow{\sigma_3} & -0120. &
\end{array}$$

Next for 1242, $(1232,1242) = (1232,1232) + (1232,0010) = 2 + 0 = 2$ and check

$$0122 \xrightarrow{\sigma_1} 1122 \xrightarrow{\sigma_2} 1222 \xrightarrow{\sigma_3} 1242 \xrightarrow{\sigma_2} 1342 \xrightarrow{\sigma_1} 2342.$$

Then finally as the W-action commutes with -1 we get the orbit for -1232, which has inner product -2 with λ.

Again, as we have accounted for all of the long roots, the result follows. ▮

RANK-2 SUBSYSTEMS

With the orbit analysis in hand we can work with representative roots to classify the $L_{\alpha,\beta}(V_\alpha)$ subsystems.

PROPOSITION 3. *Let α be a short root. If β is either a short root with $(\alpha, \beta) = 0$ or a long root with $(\alpha, \beta) = \pm 2$, then*

(i) *α and β generate a BC_2 root subsystem,*

(ii) *$L_{\alpha,\beta}(V_\alpha)$ is at most 5 dimensional*

(iii) *$L_{\alpha,\beta}(V_\alpha)$ is generated by $\{V_{\pm\gamma_0}, L(V_{\alpha_0})\}$ where γ_0 is the short non-simple root in the subsystem and α_0 is the short simple root.*

Proof. First suppose β is short. By conjugacy and Lemma 1 we may assume that $\alpha := \lambda$ and $\beta := 1110$; then $\alpha + \beta = 2342$ and $\alpha - \beta = 0122$, both long roots. In all, the set of roots generated is $\{\pm 1232, \pm 1110, \pm 2342, \pm 0122\}$, a root system of type BC_2 with simple roots 1110 and 0122. For β long, by conjugacy and Lemma 2 we may assume $\alpha := \lambda$ and $\beta := 0122$ and they generate the identical BC_2 root subsystem, proving claim (i).

Since 1110 and 0122 are the fundamental roots of the subsystem, then in either case $L_{\alpha,\beta} = \langle X_{\pm 1110}, X_{\pm 0122} \rangle \cong Sp_4(q) \cong \Omega_5(q)$. We will show that $L_{\alpha,\beta}(V_\alpha)$ is generated by a presheaf isomorphic to the natural module sheaf for C_2 (that is, $Sp_4(q)$ acting on its 4-dimensional symplectic space). We do so by checking the action of the parabolic subgroups of $L_{\alpha,\beta}$ on V_α.

There are two classes of maximal parabolic subgroups of $L_{\alpha,\beta}$ corresponding to the nodes of the diagram:

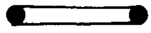
We define

$$H_{\alpha,\beta} = H \cap L_{\alpha,\beta}$$

$$P_1 = P(0122) = U_1 \cdot L_1 \quad \text{with} \quad U_1 = \langle X_{1110}, X_{1232}, X_{2342} \rangle, \quad L_1 = \langle H_{\alpha,\beta}, X_{\pm 0122} \rangle$$

$$P_2 = P(1110) = U_2 L_2 \quad \text{with} \quad U_2 = \langle X_{0122}, X_{1232}, X_{2342} \rangle, \quad L_2 = \langle H_{\alpha,\beta}, X_{\pm 1110} \rangle.$$

To determine the subspace generated by V_α under the action of each parabolic, we make use of our basic hypothesis of the sheaf construction. For the action of P_1, we note that since 0122 is in the span of π_6, then $L_1 < L(\pi_6)$ as it is defined in Part 1; our discussion there shows that $(V_{1232})^{L(\pi_6)}$ is the natural 6-dimensional symplectic module for $L(\pi_6) \cong Sp_6(q)$. In this known module we check by weight theory that $(V_{1232})^{L_1} = V_{1232} + V_{1110}$ and that this two-space is fixed by U_1. For P_2 we note that 1110 is in the span of π_1 giving that $L_2 < L(\pi_1)$. Therefore L_2 (and thus P_2) fixes V_{1232}. Finally, $P_1 \cap P_2$ is a Borel stabilizing the flag, $V_{1232} \subset V_{1232} + V_{1110}$. The resulting configuration of spaces is precisely the sheaf for the 4-dimensional symplectic module of $Sp_4(q)$:



(In particular, the high weight, $\lambda = 1232$ is the high short root of the BC_2 subsystem.) The zero-homology of that sheaf was computed by Ronan–Smith [RS, Theorems (4.1) and (4.3)]. It was determined to be at most 5-dimensional, generated by the two pairs of short root one-spaces along with possibly one additional one-space occurring in $L(V_{\alpha_0})$ where α_0 is the fundamental short root (i.e., $\alpha_0 = 1110$). By the universal property of H_0[RS, (2.3)], our subspace $L_{\alpha,\beta}(V_\alpha)$ must be a quotient of H_0. Thus, in any case $L_{\alpha,\beta}(V_\alpha)$ is generated by $V_{\pm\gamma_0}$ (γ_0 the high short root, i.e., $\gamma_0 := 1232$) and $L(V_{\alpha_0})$ ($\alpha_0 := 1110$). ∎

PROPOSITION 4. *Let α and β be short roots, with $(\alpha, \beta) = \pm 1$. Then*

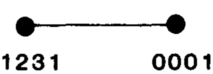
(i) *α and β generate an A_2-root subsystem*

(ii) *$L_{\alpha,\beta}(V_\alpha)$ is 8 dimensional*

(iii) *$L_{\alpha,\beta}$ is generated by $V_{\pm\gamma_0}, L(V_{\alpha_1}), L(V_{\alpha_2})$ where γ_0 is the high short root and α_1 and α_2 are fundamental for the A_2-subsystem.*

Proof. By Lemma 1 we suppose $\beta := \pm 1231$. Again we consider the subgroup $L_{\lambda,\beta} = \langle X_{\pm\lambda}, X_{\pm\beta}\rangle$. Since X_{0001} is in the group generated by X_{1232} and X_{-1231}, and $\{0001, 1231\}$ is a fundamental set of roots for the root system $\{\pm 0001, \pm 1231, \pm 1232\}$, we have $L_{\lambda,\beta} = \langle X_{1231}, X_{\pm 0001}\rangle$. Then $L_{\lambda,\beta}$ is a rank-2 Chevalley group generated by roots of equal length and thus $L_{\lambda,\beta} \cong SL_3(q)$. In this case we check that the subsystem $L_{\lambda,\beta}(V_\lambda) = \{V_\lambda^h : h \in L_{\lambda,\beta}\}$ under the action of $L_{\lambda,\beta}$ gives an isomorphic copy of the sheaf for $SL_3(q)$ acting on its adjoint module.

As in case (a), there are two classes of maximal parabolic subgroups of $L_{\lambda,\beta}$, this time corresponding to the Dynkin diagram,

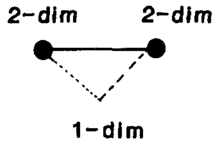


Again set

$$
\begin{aligned}
&H_{\lambda,\beta} = H \cap L_{\lambda,\beta};\\
&P_1 = P(1231) = U_1 \cdot L_1 \quad \text{with} \quad U_1 = \langle X_{0001}, X_{1232}\rangle\\
&\qquad\qquad\qquad\qquad\qquad\quad \text{and} \quad L_1 = \langle H_{\lambda,\beta}, X_{\pm 1231}\rangle,\\
&P_2 = P(0001) = U_2 \cdot L_2 \quad \text{with} \quad U_2 = \langle X_{1231}, X_{1232}\rangle\\
&\qquad\qquad\qquad\qquad\qquad\quad \text{and} \quad L_2 = \langle H_{\lambda,\beta}, X_{\pm 0001}\rangle.
\end{aligned}
$$

We see from our discussion of $V^{U(\Pi_2)}$ in Part 1 that U_2 fixes the weight spaces of 1232 and 1231, and this two-space is a natural module for $L_2 \cong SL_2(q) < L(\Pi_2) \cong SL_2(q) \times SL_3(q)$. We then can conjugate by a suitable element of W_λ which transposes 1231 and 0001, to obtain analogous statements for U_1. Lastly, $P_1 \cap P_2 = \langle H_{\lambda,\beta}, X_{1231}, X_{0001}\rangle$, a

Borel subgroup, fixes the weight space of 1232. This gives the configuration corresponding to the adjoint sheaf for type A_2:



We now appeal to the result of Smith-Völklein [SV] which gives us that $L_{\lambda,\beta}(V_\lambda)$ is 8-dimensional. In particular, the three pairs of short root one-spaces along with two additional one-spaces from $L(V_{\alpha_1})$ and $L(V_{\alpha_2})$ will generate H_0 of the sheaf (i.e., V_{1232}, $L(V_{0001})$, $L(V_{1231})$ will generate $L_{\alpha\beta}(V_\alpha)$).

The final two cases for $L_{\alpha,\beta}(V_\alpha)$ we state as corollaries to Proposition 4.

COROLLARY 5. *$L(V_\alpha)$ is three dimensional for α a short root.*

This is merely the case when $\beta = \pm\alpha$. Certainly $L_\alpha < L_{\alpha,\beta}$ for a short root β with $(\alpha, \beta) = \pm 1$. We saw above that $L_{\alpha,\beta}(V_\alpha)$ is 8 dimensional corresponding to the 8 dimensional adjoint module of $SL_3(q)$. This adjoint module corresponds to the traceless 3×3 matrices with entries in F_q. Then $L(V_\alpha)$ would correspond to 2×2 traceless matrices. In a suitable basis, L_α would be generated by

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix};$$

then V_α and $V_{-\alpha}$ would correspond to

$$\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The module generated by the action of L_α on these matrices would be generated linearly by these two along with

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

In this context, it is clear that $L(V_\alpha)$ is three dimensional.

COROLLARY 6. *For α a short root and β long with $(\alpha, \beta)=0$, then $L_{\alpha,\beta}(V_\alpha)$ is three dimensional. In fact $L_{\alpha,\beta}(V_\alpha)=L(V_\alpha)$.*

Proof. We use Lemma 2 to assume that $\alpha := 1232$ and $\beta := 1000$. Then $\alpha+\beta=2232$ and $\alpha-\beta=0232$, neither of which are roots. Then X_α and $X_{-\alpha}$ commute with X_β and $X_{-\beta}$ in G, and so $[L_\alpha, L_\beta]=1$. Also, X_{1000} and X_{-1000} lie in $L(\pi_1)$ which fixes V_{1232} (see Part 1). If we conjugate by the longest element of the Weyl group W, which interchanges α and $-\alpha$ for all roots, we conclude that X_β and $X_{-\beta}$ fix V_{-1232} also. Thus V_{1232} and V_{-1232} are fixed by L_β.

If h_α is a third one-space in $L(V_\alpha)$, then $h_\alpha=(V_\alpha)^g, g\in L_\alpha$. For any $h\in L_\beta$, $h_\alpha = (V_\alpha^g)^h = V_\alpha^{gh} = V_\alpha^{hg} = V_\alpha^g = h_\alpha$. Thus, $L(V_\alpha)^{L_\beta} = L(V_\alpha)$, so $L_{\alpha,\beta}(V_\alpha)=L(V_\alpha)$ which is three dimensional by Corollary 1. ∎

$\hat{V}=L(A)$

We now have proved that $L(A')\subset L(A)$ for A' a neighbor to A. We have already reduced this to showing that $L_{\alpha,\beta}(V_\alpha)\subset L(A)$ for any root $\beta\in\Phi(A)$. All possible cases for β are covered in Propositions 3 and 4 and Corollaries 5 and 6. In each case $L_{\alpha,\beta}(V_\alpha)$ is generated by short root subsystems contained in $L(A)$.

PROOF THAT $\dim L(A)\leqslant 26$

Since $L(A)$ is a sum of 12 $L(V_\alpha)$ subspaces, then $\dim(\hat{V})=\dim(L(A))\leqslant 36$ by Corollary 1. We can improve on this by defining $\hat{A}=\langle\{V_{\pm\alpha}:\alpha \text{ short but not simple}\}\cup\{L(V_\alpha): \alpha \text{ a simple short root}\}\rangle$. Since there are 2 short simple roots, $\dim(\hat{A})\leqslant 26$. We claim that $L(A)\subset\hat{A}$. This amounts to showing that $L(V_\gamma)\subset\hat{A}$ for any short root γ. We proceed by induction on the height of γ ($ht(\gamma)$).

(i) If $ht(\gamma)=1$ then γ is simple and therefore by definition of $\hat{A}$, $L(V_\alpha)\subset\hat{A}$.

(ii) If $ht(\gamma)>1$, it is straightforward to check that $\gamma=\alpha+\beta$ with α short and $ht(\alpha)$, $ht(\beta)<ht(\gamma)$. Since $\alpha+\beta$ is a root, the two cases in the corollaries do not arise; thus α and β generate a rank-2 system which is either BC_2 or A_2 by our propositions above.

In the BC_2 case by Proposition 3 we have that $L_{\alpha,\beta}(V_\alpha)$ is generated by $V_{\pm\gamma_0}$ and $L(V_{\alpha_0})$ where γ_0 is the high short root and α_0 is the simple short root. Since $ht(\gamma)>ht(\alpha)$, then $\gamma=\pm\gamma_0$ and $\alpha=\pm\alpha_0$. By construction, $V_{\pm\gamma}(=V_{\pm\gamma_0})\subset\hat{A}$ and by induction $L(V_\alpha)(=L(V_{\alpha_0}))\subset\hat{A}$; therefore $L_{\alpha,\beta}(V_\alpha)\subset\hat{A}$. Finally, since $L_\alpha<L_{\alpha,\beta}$ then $L(V_\gamma)\subset L_{\alpha,\beta}(V_\alpha)\subset\hat{A}$.

In the A_2 case, by Proposition 4, $L_{\alpha,\beta}(V_\alpha)$ is generated by $V_{\pm\gamma_0}$, $L(V_{\alpha_1})$ and $L(V_{\alpha_2})$ with γ_0 the high root, and α_1 and α_2 simple roots. Again, we may surmise that $\gamma = \pm\gamma_0$; while $\alpha = \pm\alpha_1$ and $\beta = \pm\alpha_2$ (reordering the simple roots if necessary). $V_{\pm\gamma} \subset \hat{A}$ by definition, and $L(V_\alpha)$, $L(V_\beta) \subset \hat{A}$ by induction. Therefore we again have $L(V_\gamma) \subset L_{\alpha,\beta}(V_\alpha) \subset \hat{A}$.

We have now established that $L(A) \subset \hat{A}$, and therefore $\hat{V} = \hat{A}$, and thus $\dim(\hat{V}) \leqslant 26$ establishing the main theorem.

Dimension of the 1-Cohomology of G

As a corollary to the main theorem, we get the 1-cohomology of G acting on V, which was first computed by Cline–Parshall–Scott [CPS]. According to Völklein [V2, Theorem 1], excluding certain cases (but not the present situation) any non-split extension of an irreducible module of a Chevalley group over the trivial module is a quotient of the 0-homology module of the fixed point sheaf. In our case, in all characteristics except 3, V is irreducible. Since $\hat{V} \cong V$, then $\mathrm{Ext}^1_{KG}(V/1) = 0$ in characteristic other than 3. For characteristic 3, if W is the 25 dimensional irreducible quotient, then $\dim \mathrm{Ext}^1_{KG}(V/1) = 1$. Since V is self-dual, we have

$$\mathrm{Ext}^1_{KG}(1/V) = 0 \qquad \text{for char.} \neq 3$$

$$\dim \mathrm{Ext}^1_{KG}(1/W) = 1 \qquad \text{for char.} = 3.$$

Finally, by a standard isomorphism we have

$$\dim(H^1_{KG}(V)) = \begin{cases} 0 & \text{char.} \neq 3 \\ 1 & \text{char.} = 3. \end{cases}$$

(This is not to be confused with equivariant cohomology.)

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